

A Conjecture about Periods in Subtraction Games

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Dedicated to Richard Guy, in celebration of his upcoming 100th birthday.

Abstract

We make a conjecture that characterizes the periods of the nim values in subtraction games with subtraction set of size 3.

1 Introduction

Richard J. Nowakowski has, for many years, maintained a document of unsolved problems in combinatorial games [2]; subtraction games are the very first problem discussed. Although they are fundamentally important in the realm of combinatorial games, they have been surprisingly challenging to completely analyze.

A subtraction game $S(s_1, s_2, s_3, \dots)$ is played in much the same way as Nim, but the number of beans that can be removed from a heap during a player's turn is limited to the subtract set $\{s_1, s_2, s_3, \dots\}$.

To follow the notation of Nowakowski, if the nim value of a heap of size h in a subtraction game is written as n_h , then the analogous nim sequence for the subtraction game is $n_0 n_1 n_2 n_3 n_4 \dots$. For many examples of these nim sequences for subtraction games, see [1, pp. 84–85, Table 1]. It is well known that, if the subtraction set has a finite size, then the analogous nim sequence eventually is periodic, but the periods of the nim values remain largely mysterious.

Mark Paulhus and Alex Fink have derived values of the periods in two cases, for subtraction sets of size 3, namely, in the case where $s_1 = 1$ and s_2, s_3 are arbitrary, and in the case where $s_1 < s_2 < s_3 < 32$ (see [2]). To the best of the author's knowledge, in the latter case $s_1 < s_2 < s_3 < 32$, Paulhus and Fink did not have a formula that characterizes the periods, but rather, they derived the values of the periods in these $\binom{32}{3}$ cases (without deriving a general formula to characterize the periods).

2 Characterization of Periods

In the present manuscript, we give a characterization of the possible values of the periods in subtraction games with subtraction sets of size 3. We use “gcd” as shorthand for “greatest common divisor.”

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Conjecture 1. Consider a subtraction game $S(s_1, s_2, s_3)$ with $s_1 < s_2 < s_3$, in which the nim sequence $n_0 n_1 n_2 n_3 n_4 \dots$ eventually has period p .

Case I. If $s_3 = s_1 + s_2$, define $0 \leq j < 2s_1$ so that $s_2 - s_1 \equiv j \pmod{2s_1}$. Then the period p can be precisely characterized as follows:

$$p = \begin{cases} s_2 + s_3 - j & \text{if } 0 \leq j < s_1, \\ (s_1)(s_2 + s_3 + j - 2s_1) / \gcd(s_1, 2s_1 - j) & \text{if } s_1 \leq j < 2s_1. \end{cases}$$

Case II. If $s_3 \neq s_1 + s_2$, then p has one of seven potential values, which can be characterized as follows: The period p is a divisor of **at least one** of the numbers $s_i + s_j$ for $1 \leq i < j \leq 3$, and moreover, the period p is **exactly the gcd of all such terms**, i.e.,

$$p = \gcd_{(i,j) \in \mathcal{G}} (s_i + s_j),$$

where \mathcal{G} is the set of pairs (i, j) such that $s_i + s_j$ is a multiple of p .

The conjecture characterizes the period p of every subtraction game $S(s_1, s_2, s_3)$ with subtraction set of size 3. I have verified the conjecture in all $\binom{4096}{3} = 11,444,858,880$ cases in which $1 \leq s_1 < s_2 < s_3 \leq 4096$.

3 Acknowledgements

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References

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